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The complete Faddeev-Jackiw treatment of the $U_{EM}(1)$ gauged SU(2) WZW model [1]

J.E.Paschalis and P.I.Porfyriadis

Department of Theoretical Physics, University of Thessaloniki,
GR 54006 , Thessaloniki , Greece

PASCHALIS@OLYMP.CCF.AUTH.GR
PORFYRIADIS@OLYMP.CCF.AUTH.GR

Abstract

The two flavour, four dimensional WZW model coupled to electromagnetism, is treated as a constraint system in the context of the Faddeev-Jackiw approach. No approximation is made. Detailed exposition of the calculations is given. Solution of the constraints followed by proper Darboux's transformations leads to an unconstrained Coulomb-gauge Lagrangian density.

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1 Introduction

The basic feature of the Faddeev-Jackiw (FJ) approach [1, 2] to constrained systems is the fact that it treats all the constraints on an equal basics. There is no distinction between first and second class constraints as in the case of the Dirac quantization procedure [3, 4]. Also constraints involving canonical momenta p_i conjugate to certain variables q_i , whose velocities $\dot{q}_i$ occur lineary or do not occur at all in the Lagrangian, are absent in the FJ approach contrary to the Dirac procedure.

The FJ approach is based on the Darboux's theorem [2, 5], according to which an arbitrary vector potential $a_i(\xi)$, whose associated field strength $f_{ij}(\xi)$ is non-singular can be written apart from a gauge transformation in the form

$$a_i(\xi) = \frac{1}{2} Q^k(\xi) \omega_{kl} \frac{\partial Q^l(\xi)}{\partial \xi^i} \quad . \tag{1}$$

and the field strength as

$$f_{ij}(\xi) = \frac{\partial Q^k(\xi)}{\partial \xi^i} \omega_{kl} \frac{\partial Q^l(\xi)}{\partial \xi^j} \quad . \tag{2}$$

where $Q^i(\xi)$ are the new (Darboux transformed) coordinates and $[\omega_{ij}]$ is a constant, non-singular antisymmetric matrix. In this case the associated canonical one form $a_i(\xi) d\xi^i$ is called diagonal.

The starting point of the FJ approach is a Lagrangian first order in the time derivatives. Any conventional, second order in time derivatives Lagrangian can be written as a first order expression by properly enlarging its configuration space so that it includes the conjugate momenta of the coordinate variables. For a constrained system described by the Lagrangian

$$\mathcal{L}(\xi, \dot{\xi}) = a_i(\xi) \dot{\xi}^i - H(\xi) \quad , \quad i = 1, ..., N \tag{3}$$

the field strength $f_{ij} = \frac{\partial}{\partial \xi_i} a_j(\xi) - \frac{\partial}{\partial \xi_j} a_i(\xi)$ is a singular $N \times N$ matrix. The Darboux's transformations can be applied to the maximal non-singular submatrix of f_{ij}. The Lagrangian tranforms into

$$\mathcal{L}(Q, \dot{Q}, z) = \frac{1}{2} Q^k(\xi) \omega_{kl} \dot{Q}^l(\xi) - H(Q, z) \quad , \quad k, l = 1, ..., 2n \tag{4}$$

where by z we denote the coordinates left unchanged. Then by using the Euler-Lagrange equations we solve for as many z's as possible in terms of other coordinates and we obtain

$$\mathcal{L}(Q, \dot{Q}, z) = \frac{1}{2} Q^k(\xi) \omega_{kl} \dot{Q}^l(\xi) - H'(Q) - z^m \Phi_m(Q) \quad , \tag{5}$$

where Φ_m are the true constraints of the system. Next we solve the constraint equations $\Phi_m(Q) = 0$ for some of the Q's and substitute back in (5). We end up with a Lagrangian with the same form as the initial (3) and with fewer dynamical variables. The whole procedure is then repeated again until we are left with an unconstrained Lagrangian, with reduced coordinate space and with diagonal canonical one-form. In the case that

the constraints cannot be solved due to technical difficulties one can return to the Dirac procedure.

The equivalence of the FJ approach to the Dirac method is discussed in [11, 12]. Its extension in superspace is given in [13, 14]. Application of the approach to the light-cone quantum field theory is given in [15, 16], to non-abelian systems in [17] to hidden symmetries in [18] and to self-dual fields in [19]. Other works related to the FJ approach are given in [20, 21, 22].

2 The $U_{EM}(1)$ gauged SU(2) WZW model in the FJ framework

The model [6, 7, 8, 9] describes the low energy interactions of pions and photons including those related to the QCD anomalies. It posesses the symmetries of electromagnetism and those relevant to QCD and has no extra symmetries. The effective action is given by

$$\begin{aligned}
\Gamma_{eff}(U,A_\mu) &= \Gamma_{EM}(A_\mu)+\Gamma_\sigma(U,A_\mu)+\Gamma_{WZW}(U,A_\mu) \quad , \qquad (6)\\
\Gamma_{EM}(A_\mu) &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \quad ,\\
\Gamma_\sigma(U,A_\mu) &= -\frac{f_\pi^2}{16}\int d^4x \mathrm{tr}\,(R_\mu R^\mu)\\
&= -\frac{f_\pi^2}{16}\int d^4x \mathrm{tr}\,(r_\mu r^\mu)+\frac{if_\pi^2 e}{8}\int d^4x A_\mu \mathrm{tr}\,[Q(r_\mu - l_\mu)]\\
&+\frac{f_\pi^2 e^2}{8}\int d^4x A_\mu A^\mu \mathrm{tr}\,(Q^2-U^\dagger Q U Q) \quad ,\\
\Gamma_{WZW}(U,A_\mu) &= -\frac{N_c e}{48\pi^2}\int d^4x \epsilon^{\mu\nu\alpha\beta}A_\mu \mathrm{tr}\,[Q(r_\nu r_\alpha r_\beta + l_\nu l_\alpha l_\beta)]\\
&+\frac{iN_c e^2}{24\pi^2}\int d^4x \epsilon^{\mu\nu\alpha\beta}A_\mu(\partial_\nu A_\alpha)\mathrm{tr}\,[Q^2(r_\beta+l_\beta)\\
&+\frac{1}{2}QU^\dagger Q U r_\beta + \frac{1}{2}QUQU^\dagger l_\beta] \quad ,
\end{aligned}$$

where

$$\begin{aligned}
&U = \exp(2i\theta_i\tau_i/f_\pi) \quad , \quad r_\mu = U^\dagger\partial_\mu U \quad , \quad R_\mu = U^\dagger D_\mu U \quad ,\\
&l_\mu = (\partial_\mu U)U^\dagger \quad , \quad L_\mu = (D_\mu U)U^\dagger \quad .
\end{aligned}$$

See Appendix for notation.

In the SU(3) case the effective action should include one more term

$$\Gamma_{WZW}(U) = -\frac{iN_c}{240\pi^2}\int d^5x \epsilon^{ijklm}\mathrm{tr}\,(l_i l_j l_k l_l l_m) \quad , \qquad (7)$$

which describes processes like $K^+K^- \to \pi^+\pi^0\pi^-$ (in the low energy regime) which respect the combined discrete operations $U(\mathbf{x},\mathrm{t}) \to \mathrm{U}(-\mathbf{x},\mathrm{t})$, $\mathrm{U}(\mathbf{x},\mathrm{t}) \to \mathrm{U}^\dagger(\mathbf{x},\mathrm{t})$ but not each

one seperately. In the SU(2) case this term vanishes identically. Nevertheless the term $\Gamma_{WZW}(U, A_\mu)$ which comes from gauging (7) is present and describes processes like $\pi^0 \to 2\gamma$, $\gamma \to 3\pi$, $2\gamma \to 3\pi$ etc. related to the QCD anomaly.

In [10] the U field was expanded in powers of the Goldstone boson fields θ_a. Keeping terms up to second and third order we obtained Lagrangian densities with only one true constraint, the one which is multiplied by A_0 (scalar potential). Then after solving the equation of the constraint and performing the proper Darboux's transformations the longitudinal part of the vector potential was cancelled out and we were left with an unconstrained Coulomb-gauge Lagrangian density. This is exactly what happens also in the case of spinor electrodynamics [1].

In this work no expansion is made. Instead a complete treatment is given to the model as a constrained system in the context of the FJ formalism. The following parametrization of the U field is used

$$U = \cos(2|\theta|/f_\pi) + i\tau_i\hat{\theta}_i \sin(2|\theta|/f_\pi) = \phi_0(2|\theta|/f_\pi) + i\tau_i\phi_i(2|\theta|/f_\pi) \ , \tag{8}$$

where the real fields ϕ_0 , ϕ_i , $i = 1, 2, 3$ are subject to the unitarity constraint

$$\phi_0^2 + \phi_i^2 = 1 \tag{9}$$

The Lagrangian density of the model is written as follows in terms of the ϕ-fields

$$\begin{aligned}
\mathcal{L}_{eff} &= \mathcal{L}_{EM} + \mathcal{L}_\sigma + \mathcal{L}_{WZW}, \qquad (10)\\
\mathcal{L}_{EM} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ ,\\
\mathcal{L}_\sigma &= \frac{f_\pi^2}{8}\partial_\mu\phi_0\partial^\mu\phi_0 + \frac{f_\pi^2}{8}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{f_\pi^2 e}{4}A^\mu(\phi_2\partial_\mu\phi_1 - \phi_1\partial_\mu\phi_2) + \frac{f_\pi^2 e^2}{8}A_\mu A^\mu(\phi_1^2 + \phi_2^2) \ ,\\
\mathcal{L}_{WZW} &= -\frac{2Ne}{\phi_0}\epsilon^{\mu\nu\alpha\beta}A_\mu\partial_\nu\phi_1\partial_\alpha\phi_2\partial_\beta\phi_3\\
&\quad + 2Ne^2\phi_3\epsilon^{\mu\nu\alpha\beta}A_\mu(\partial_\nu A_\alpha)\partial_\beta\phi_0 - Ne^2\phi_0\phi_3\epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)(\partial_\alpha A_\beta) \ ,
\end{aligned}$$

where $N = \frac{N_c}{24\pi^2}$. The canonical momenta conjugate to ϕ_0 , ϕ_i are given by

$$\begin{aligned}
p_0 &= \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\phi}_0} = \frac{f_\pi^2}{4}\dot{\phi}_0 - 2Ne^2(\mathbf{A}\cdot\mathbf{B})\phi_3 \ ,\\
p_1 &= \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\phi}_1} = \frac{f_\pi^2}{4}\dot{\phi}_1 + \frac{f_\pi^2 e}{4}A_0\phi_2 - \frac{2Ne}{\phi_0}(\nabla\phi_2\times\nabla\phi_3)\cdot\mathbf{A} \ ,\\
p_2 &= \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\phi}_2} = \frac{f_\pi^2}{4}\dot{\phi}_2 - \frac{f_\pi^2 e}{4}A_0\phi_1 + \frac{2Ne}{\phi_0}(\nabla\phi_1\times\nabla\phi_3)\cdot\mathbf{A} \ ,\\
p_3 &= \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\phi}_3} = \frac{f_\pi^2}{4}\dot{\phi}_3 - \frac{2Ne}{\phi_0}(\nabla\phi_1\times\nabla\phi_2)\cdot\mathbf{A} \ .
\end{aligned} \tag{11}$$

In the enlarged configuration space with coordinates $-\boldsymbol{\pi}$, $\mathbf{A}$, p_0 , ϕ_0 , $\mathrm{p_i}$, ϕ_i , $\mathrm{i} = 1, 2, 3$ the effective Lagrangian density (10) is written as an expression first order in time

derivatives as follows

$$\mathcal{L}_{eff} = -\boldsymbol{\pi}\cdot\dot{\mathbf{A}} + \mathrm{p_0}\dot{\phi}_0 + \mathrm{p_i}\dot{\phi}_\mathrm{i} - \mathrm{H}^0 - \mathrm{A}_0(\rho_\sigma + \rho_\mathrm{w} - \nabla\cdot\boldsymbol{\pi}) \quad , \tag{12}$$

$$\begin{aligned}
H^0 =& \frac{1}{2}[\boldsymbol{\pi}^2 + \mathbf{B}^2] \\
&+ \frac{f_\pi^2}{8}(\nabla\phi_0)^2 + \frac{f_\pi^2}{8}(\nabla\phi_i)^2 + \frac{f_\pi^2 e}{4}\mathbf{A}\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) + \frac{\mathrm{f}_\pi^2\mathrm{e}^2}{8}\mathbf{A}^2(\phi_1^2 + \phi_2^2) \\
&+ \frac{2}{f_\pi^2}[p_i + \frac{Ne}{\phi_0}\epsilon_{ijk}\mathbf{A}\cdot(\nabla\phi_\mathrm{j}\times\nabla\phi_\mathrm{k})]^2 + \frac{2}{\mathrm{f}_\pi^2}[\mathrm{p}_0 + 2\mathrm{Ne}^2\phi_3(\mathbf{A}\cdot\mathbf{B})]^2 \\
&- 2Ne^2\phi_3(\mathbf{A}\times\boldsymbol{\pi})\cdot\nabla\phi_0 - 2\mathrm{Ne}^2\phi_0\phi_3(\boldsymbol{\pi}\cdot\mathbf{B}) \quad , \\
\rho_\sigma =& e(p_2\phi_1 - p_1\phi_2) \quad , \\
\rho_\mathrm{w} =& 2Ne^2\nabla\phi_0\cdot\nabla\times(\phi_3\mathbf{A}) + \frac{2\mathrm{Ne}}{\phi_0}(\nabla\phi_1\times\nabla\phi_2)\cdot\nabla\phi_3 \quad .
\end{aligned}$$

We see that the canonical one form in (12) can be written apart from a total time derivative, in the diagonal form defined in (1). Taking the time derivative of the unitarity constraint (9) and substituting the time derivatives of the ϕ-fields from (11) we obtain the following secondary constraint

$$p_0\phi_0 + p_i\phi_i + \frac{Ne}{\phi_0}\epsilon_{ijk}\phi_i(\nabla\phi_j\times\nabla\phi_k)\cdot\mathbf{A} + 2\mathrm{Ne}^2\phi_0\phi_3(\mathbf{A}\cdot\mathbf{B}) = 0 \tag{13}$$

Setting the time derivative of (13) equal to zero does not give any new constraint. One more constraint appears in (12) multiplied by A_0

$$\nabla\cdot\boldsymbol{\pi} - \rho_\sigma - \rho_\mathrm{w} = 0 \tag{14}$$

whose time derivative also does not add any new one to the model. So finally we have three true second class constraints given by the equations (9),(13),(14). In order to solve (14) we decompose the electric field $\boldsymbol{\pi}$ and the vector potential $\mathbf{A}$ into transverse and longitudinal components

$$\mathbf{A}^\mathrm{T} = \mathbf{A} - \nabla\mathrm{A}^{\mathrm{L}'} \quad , \qquad \mathbf{A}^\mathrm{L} = \nabla\mathrm{A}^{\mathrm{L}'} \quad , \qquad \mathrm{A}^{\mathrm{L}'} = \frac{1}{\nabla^2}(\nabla\cdot\mathbf{A}) \quad ,$$

$$\boldsymbol{\pi}^T = \boldsymbol{\pi} - \frac{\nabla}{\nabla^2}\pi^{L'} \quad , \qquad \boldsymbol{\pi}^L = \frac{\nabla}{\nabla^2}\pi^{L'} \quad , \qquad \pi^{L'} = \nabla\cdot\boldsymbol{\pi} \quad .$$

Then (14) implies that

$$\boldsymbol{\pi}^L = \frac{\nabla}{\nabla^2}(\rho_\sigma + \rho_\mathrm{w}) \quad . \tag{15}$$

We substitute into (12) and we obtain, apart from a total spatial derivative, the following expression for the effective Lagrangian density

$$\mathcal{L}_{eff} = -\boldsymbol{\pi}^T\cdot\dot{\mathbf{A}}^\mathrm{T} + \mathrm{p_0}\dot{\phi}_0 + \mathrm{p_i}\dot{\phi}_\mathrm{i} + (\rho_\sigma + \rho_\mathrm{w})\frac{\nabla}{\nabla^2}\cdot\dot{\mathbf{A}}^\mathrm{L} - \mathrm{H}^0 \quad , \tag{16}$$

where the expression for H^0 is given in (12) with $\boldsymbol{\pi}^L$ substituted by (15). We see that $\mathbf{A}^{\mathrm{L}}$ enters the canonical one form in an uncanonical way. A partial diagonalization can be achieved by performing the following Darboux's transformations

$$
\begin{aligned}
&p_1 \to p_1 \cos\alpha + p_2 \sin\alpha \ , \qquad \phi_1 \to \phi_1 \cos\alpha + \phi_2 \sin\alpha \ , \\
&p_2 \to p_2 \cos\alpha - p_1 \sin\alpha \ , \qquad \phi_2 \to \phi_2 \cos\alpha - \phi_1 \sin\alpha \ ,
\end{aligned} \tag{17}
$$

where $\alpha = e\frac{\nabla}{\nabla^2} \cdot \mathbf{A}^{\mathrm{L}}$

Then the effective Lagrangian density acquires the form

$$
\mathcal{L}_{eff} = -\boldsymbol{\pi}^T \cdot \dot{\mathbf{A}}^{\mathrm{T}} + \mathrm{p_0}\dot{\phi}_0 + \mathrm{p_i}\dot{\phi}_{\mathrm{i}} + \rho_{\mathrm{w}} \frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}^{\mathrm{L}} - \mathrm{H}^0 \ . \tag{18}
$$

We have redefined H^0 after having performed the Darboux's transformations (17) into its previous expression in (16). Also ρ_{w} given initially in (12) becomes

$$
\rho_{\mathrm{w}} = 2Ne^2 \nabla\phi_0 \cdot \nabla \times (\phi_3 \mathbf{A}^{\mathrm{T}}) + \frac{2\mathrm{Ne}}{\phi_0} (\nabla\phi_1 \times \nabla\phi_2) \cdot \nabla\phi_3 \ .
$$

We see that $\mathbf{A}^{\mathrm{L}}$ cancels out. This is important for the next step which is the elimination of $\rho_{\mathrm{w}}\frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}^{\mathrm{L}}$ from the canonical one-form in (18). Apart from total derivatives we have

$$
\begin{aligned}
\mathcal{L}_{eff} = &-[\boldsymbol{\pi}^T - 2Ne^2(\nabla\phi_0 \times \mathbf{A}^{\mathrm{L}})\phi_3] \cdot \dot{\mathbf{A}}^{\mathrm{T}} + [\mathrm{p_0} + 2\mathrm{Ne^2}\mathbf{A}^{\mathrm{L}} \cdot \nabla \times (\phi_3 \mathbf{A}^{\mathrm{T}})]\dot{\phi}_0 \\
&+ [p_i + \frac{Ne}{\phi_0}\epsilon_{ijk}(\nabla\phi_j \times \nabla\phi_k) \cdot \mathbf{A}^{\mathrm{L}} + 2\mathrm{Ne^2}\delta_{\mathrm{i3}}(\nabla\phi_0 \times \mathbf{A}^{\mathrm{L}}) \cdot \mathbf{A}^{\mathrm{T}}]\dot{\phi}_{\mathbf{i}} - \mathrm{H}^0
\end{aligned} \tag{19}
$$

We perform the second set of Darboux's transformations

$$
\begin{aligned}
&\boldsymbol{\pi}^T \to \boldsymbol{\pi} + 2Ne^2(\nabla\phi_0 \times \mathbf{A}^{\mathrm{L}})\phi_3 \ , \\
&p_0 \to p_0 - 2Ne^2 \mathbf{A}^{\mathrm{L}} \cdot \nabla \times (\phi_3 \mathbf{A}^{\mathrm{T}}) \ , \\
&p_i \to p_i - \frac{Ne}{\phi_0}\epsilon_{ijk}(\nabla\phi_j \times \nabla\phi_k) \cdot \mathbf{A}^{\mathrm{L}} - 2\mathrm{Ne^2}\delta_{\mathrm{i3}}(\nabla\phi_0 \times \mathbf{A}^{\mathrm{L}}) \cdot \mathbf{A}^{\mathrm{T}} \ , \quad \mathrm{i} = 1, 2, 3
\end{aligned} \tag{20}
$$

which diagonalizes completely the canonical one-form in (19). We note that the Darboux tranformed field $\boldsymbol{\pi}$ is no longer tranverse. So the full expression for the effective Lagrangian density (12) transforms as follows under the first two sets of Darboux's transformations (11) and (20)

$$
\mathcal{L}_{eff} = -\boldsymbol{\pi} \cdot \dot{\mathbf{A}}^{\mathrm{T}} + \mathrm{p_0}\dot{\phi}_0 + \mathrm{p_i}\dot{\phi}_{\mathrm{i}} - \mathrm{H}^0 - \mathrm{H}(\mathbf{A}^{\mathrm{L}}) \ , \tag{21}
$$

$$
\begin{aligned}
H^0 = &\frac{1}{2}[\boldsymbol{\pi}^2 - \rho^* \frac{1}{\nabla^2}\rho^* + \mathbf{B^2}] \\
&+ \frac{f_\pi^2}{8}(\nabla\phi_0)^2 + \frac{f_\pi^2}{8}(\nabla\phi_i)^2 + \frac{f_\pi^2 e}{4}\mathbf{A}^{\mathrm{T}} \cdot (\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) + \frac{\mathrm{f}_\pi^2\mathrm{e}^2}{8}(\mathbf{A}^{\mathrm{T}})^2(\phi_1^2 + \phi_2^2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{f_\pi^2}[p_i+\frac{Ne}{\phi_0}\epsilon_{ijk}\mathbf{A}^{\rm T}\cdot(\nabla\phi_{\rm j}\times\nabla\phi_{\rm k})]^2+\frac{2}{{\rm f}_\pi^2}[{\rm p}_0+2{\rm Ne}^2\phi_3(\mathbf{A}^{\rm T}\cdot\mathbf{B})]^2\\
&-2Ne^2\phi_3(\boldsymbol{\pi}+\frac{\nabla}{\nabla^2}\rho^*)\cdot\nabla\times(\phi_0\mathbf{A}^{\rm T})-\nabla\cdot\boldsymbol{\pi}\frac{1}{\nabla^2}\rho^*\ \ ,\\
H(\mathbf{A}^{\rm L}) &= -2N^2e^4\phi_3^2(\nabla\phi_0\times\mathbf{A}^{\rm L})^2-4{\rm N}^2{\rm e}^4\phi_3^2(\nabla\phi_0\times\mathbf{A}^{\rm L})\cdot\nabla\times(\phi_0\mathbf{A}^{\rm T})\\
&-\frac{8N^2e^4}{f_\pi^2\phi_0^2}[(\nabla\phi_3\times\mathbf{A}^{\rm T})\cdot\mathbf{A}^{\rm L}]^2\ \ ,\\
\rho^* &= \rho_\sigma+\rho_{\rm w}-\nabla\cdot\boldsymbol{\pi}\ \ ,\\
\rho_\sigma &= e(p_2\phi_1-p_1\phi_2)\ \ ,\\
\rho_{\rm w} &= 2Ne^2\nabla\phi_0\cdot\nabla\times(\phi_3\mathbf{A}^{\rm T})+\frac{2{\rm Ne}}{\phi_0}(\nabla\phi_1\times\nabla\phi_2)\cdot\nabla\phi_3\ \ .
\end{aligned}
$$

In order to obtain (21) we made use of all three constraints. It is interesting to note that the unitarity constraint (9) does not change under the first two sets of Darboux's transformations while the second one in (13) transform into

$$
p_0\phi_0+p_i\phi_i+\frac{Ne}{\phi_0}\epsilon_{ijk}\phi_i(\nabla\phi_j\times\nabla\phi_k)\cdot\mathbf{A}^{\rm T}+2{\rm Ne}^2\phi_0\phi_3(\mathbf{A}^{\rm T}\cdot\mathbf{B})-\frac{2{\rm Ne}^2}{\phi_0}(\mathbf{A}^{\rm T}\times\mathbf{A}^{\rm L})\cdot\nabla\phi_3=0
$$

and it is this form of the constraint that was actually used. We see in (21) that although $\mathbf{A}^{\rm L}$ disappears from the canonical one-form, still appears in the Hamiltonian density and actually that part which comes from the Wess-Zumino term, contrary to what happens in [10] where the U-field was expanded in series of powers of the pion fields.

The ϕ-fields are not independent due to the unitarity constraint (9). So we use (9) to express $\dot{\phi}_0$ in terms of the other ϕ-fields. Replasing into (21) we see that the canonical one-form loses its diagonal form

$$
\mathcal{L}_{eff}=-\boldsymbol{\pi}\cdot\dot{\mathbf{A}}^{\rm T}+({\rm p_i}-\frac{{\rm p}_0}{\phi_0}\phi_{\rm i})\dot{\phi}_{\rm i}-{\rm H}^0-{\rm H}(\mathbf{A}^{\rm L})\ \ . \tag{22}
$$

Rediagonalization can be realized by performing the following set of Darboux's transformations

$$
p_i\to p_i+\frac{p_0}{\phi_0}\phi_i\ \ ,\quad i=1,2,3
$$

Next we decompose $\boldsymbol{\pi}$ into longitudinal and transverse components. We obtain after cancellations the following expression for the effective Lagrangian density

$$
\begin{aligned}
\mathcal{L}_{eff} &= -\boldsymbol{\pi}^T\cdot\dot{\mathbf{A}}^{\rm T}+{\rm p_i}\dot{\phi}_{\rm i}-{\rm H}^0-{\rm H}(\mathbf{A}^{\rm L})-{\rm H}({\rm p}_0)\ \ , \qquad (23)\\
H^0 &= \frac{1}{2}[(\boldsymbol{\pi}^T)^2-\rho\frac{1}{\nabla^2}\rho+\mathbf{B^2}]\\
&+\frac{f_\pi^2}{8}(\nabla\phi_0)^2+\frac{f_\pi^2}{8}(\nabla\phi_i)^2+\frac{f_\pi^2e}{4}\mathbf{A}^{\rm T}\cdot(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)+\frac{{\rm f}_\pi^2{\rm e}^2}{8}(\mathbf{A}^{\rm T})^2(\phi_1^2+\phi_2^2)\\
&+\frac{2}{f_\pi^2}[p_i+\frac{Ne}{\phi_0}\epsilon_{ijk}\mathbf{A}^{\rm T}\cdot(\nabla\phi_{\rm j}\times\nabla\phi_{\rm k})]^2+\frac{8{\rm N}^2{\rm e}^4}{{\rm f}_\pi^2}\phi_3^2(\mathbf{A}^{\rm T}\cdot\mathbf{B})^2
\end{aligned}
$$

$$
\begin{aligned}
&- 2Ne^2\phi_3(\boldsymbol{\pi}^T + \frac{\nabla}{\nabla^2}\rho)\cdot\nabla\times(\phi_0\mathbf{A}^{\mathrm{T}}) \ \ , \\
H(p_0) =& \frac{2}{f_\pi^2}\frac{p_0^2}{\phi_0^2} + \frac{4}{f_\pi^2}\frac{p_0}{\phi_0}[p_i\phi_i + \frac{Ne}{\phi_0}\epsilon_{ijk}\phi_i(\nabla\phi_j\times\nabla\phi_k)\cdot\mathbf{A}^{\mathrm{T}} + 2\mathrm{Ne}^2\phi_0\phi_3(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B})] \ \ , \\
\rho =& \rho_\sigma + \rho_{\mathrm{w}} \ \ ,
\end{aligned}
$$

where $H(\mathbf{A}^{\mathrm{L}})$, ρ_σ , ρ_{w} is given in (21). The fact that the variables $\mathbf{A}^{\mathrm{L}}$ and p_0 which are absent in the canonical one-form still occur in the Hamiltonian leads to a new set of constraint equations.

$$
\frac{\partial H(\mathbf{A}^{\mathrm{L}})}{\partial\mathbf{A}^{\mathrm{L}}} = 0 \quad , \quad \frac{\partial H(p_0)}{\partial p_0} = 0 \tag{24}
$$

the solutions of which are given by

$$
\begin{aligned}
\mathbf{A}^{\mathrm{L}} =& -\mathbf{A}^{\mathrm{T}} + \frac{\phi_0}{(\nabla\phi_0)^2}\frac{(\mathbf{A}^{\mathrm{T}}\times\nabla\phi_3)\times[\nabla\phi_0\times(\mathbf{B}\times\nabla\phi_0)]}{(\nabla\phi_3\times\nabla\phi_0)\cdot\mathbf{A}^{\mathrm{T}}} \ \ , \qquad (25) \\
p_0 =& -[p_i\phi_i + \frac{Ne}{\phi_0}\epsilon_{ijk}\phi_i(\nabla\phi_j\times\nabla\phi_k)\cdot\mathbf{A}^{\mathrm{T}} + 2\mathrm{Ne}^2\phi_0\phi_3(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B})]\phi_0 \ \ .
\end{aligned}
$$

Note also that

$$
det\left[\frac{\partial H(\mathbf{A}^{\mathrm{L}})}{\partial A_i^L\partial A_j^L}\right] = -\frac{256N^6e^{12}}{f_\pi^2}\frac{\phi_3^4}{\phi_0^2}(\nabla\phi_0)^2[\nabla\phi_0\cdot(\nabla\phi_3\times\mathbf{A}^{\mathrm{T}})]^2 \neq 0
$$

So finally after substituting (25) into (23) we are left with an unconstrained Lagrangian density with reduced coordinate space and diagonal canonical one-form given by

$$
\begin{aligned}
\mathcal{L}_{eff} =& -\boldsymbol{\pi}^T\cdot\dot{\mathbf{A}}^{\mathrm{T}} + \mathrm{p_i}\dot{\phi}_{\mathrm{i}} - \mathrm{H_C} \ \ , \qquad (26) \\
H_C =& \frac{2}{f_\pi^2}[p_i + \frac{Ne}{\phi_0}\epsilon_{ijk}\mathbf{A}^{\mathrm{T}}\cdot(\nabla\phi_{\mathrm{j}}\times\nabla\phi_{\mathrm{k}})]^2 + \frac{8\mathrm{N}^2\mathrm{e}^4}{\mathrm{f}_\pi^2}\phi_3^2(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B})^2 \\
&- \frac{2}{f_\pi^2}[p_i\phi_i + \frac{Ne}{\phi_0}\epsilon_{ijk}\phi_i(\nabla\phi_j\times\nabla\phi_k)\cdot\mathbf{A}^{\mathrm{T}} + 2\mathrm{Ne}^2\phi_0\phi_3(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B})]^2 \\
&+ \ \frac{1}{2}[(\boldsymbol{\pi}^T)^2 + \mathbf{B}^2 - \rho\frac{1}{\nabla^2}\rho + \frac{\mathrm{f}_\pi^2}{4}(\nabla\phi_0)^2 + \frac{\mathrm{f}_\pi^2}{4}(\nabla\phi_{\mathrm{i}})^2] \\
&+ \frac{f_\pi^2 e}{4}\mathbf{A}^{\mathrm{T}}\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) + \frac{\mathrm{f}_\pi^2\mathrm{e}^2}{8}(\mathbf{A}^{\mathrm{T}})^2(\phi_1^2 + \phi_2^2) \\
&- \ 2Ne^2\phi_3(\boldsymbol{\pi}^T + \frac{\nabla}{\nabla^2}\rho)\cdot\nabla\times(\phi_0\mathbf{A}^{\mathrm{T}}) + 2\mathrm{N}^2\mathrm{e}^4[\phi_3\nabla\times(\phi_0\mathbf{A}^{\mathrm{T}})]^2 \\
&- 2N^2e^4\frac{(\phi_0\phi_3\nabla\phi_0\cdot\mathbf{B})^2}{(\nabla\phi_0)^2} \ \ ,
\end{aligned}
$$

where $\phi_0 = (1-\phi_i^2)^{\frac{1}{2}}$. We see that only the physical transverse components of the vector potential enter in the expression of the Lagrangian density. (Note that $\mathbf{B} = \nabla\times\mathbf{A}^{\mathrm{T}}$).

3 Conclusion

The two-flavour Wess-Zumino-Witten model coupled to electromagnetism is treated as a constrained system in the context of the Faddeev-Jackiw formalism. The treatment is complete since no expansion is used as in (10) where the U field is expanded in series of powers of the pion fields. The system has three true second class constraints. We write the Lagrangian density as an expression first order in time derivatives of the dynamical variables. Then by solving the constraints for some of the variables, substituting back into the Lagrangian density and performing Darboux's transformations we obtain a canonical expression with reduced coordinate but with the unphysical $\mathbf{A}^{\mathrm{L}}$ still occuring in the Hamiltonian density and specifically in the part which comes from the Wess-Zumino term. This is to be contrasted with the former case (10) where $\mathbf{A}^{\mathrm{L}}$ cancels out exactly when term up to second and third order in the Goldstone boson fields are kept. The existence of $\mathbf{A}^{\mathrm{L}}$ and p_0 in the Hamiltonian and not in the canonical one-form leads to a new set of constraint equations the solution of which fixes the value of $\mathbf{A}^{\mathrm{L}}$ and p_0. Finally we end up with unconstrained Coulomb-gauge canonical Lagrangian density (25) whose coordinate space consists of ϕ_i $(i = 1, 2, 3)$, $\mathbf{A}^{\mathrm{T}}$ and their conjugate momenta.

4 Appendix

Our metric is $g_{\mu\nu} = diag(1, -1, -1, -1)$, $Q = diag(2/3, -1/3)$ is the charge matrix, $D_\mu = \partial_\mu + ieA_\mu[Q,\ ]$ denotes the covariant derivative. By τ_i , $i = 1, 2, 3$ we denote Pauli matrices. We choose $e > 0$ so that the electric charge of the electron is $-e$. We define $\epsilon^{0123} = 1$. By $\boldsymbol{\pi}$ we denote the electric field $\mathbf{E}$. Summation over repeated indices is understood.

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